Math 249 Wednesday, April 29

HL pols $H_\mu \longrightarrow \tilde{H}_\mu(x;q) = q^{n(\mu)} H_\mu(x;q^{-1})$ $\qquad n(\mu) = \sum_i (i-1)\mu_i$

$K_{(n),\mu} = \langle s_{(n)}, H_\mu \rangle = \langle h_n, H_\mu \rangle = H_\mu[1;q^{-1}]$
$n = |\mu|$ $\qquad = q^{n(\mu)}$

2	2	
1	1	1
0	0	0

$\mu(3,3,2)$
$n(\mu) = 7$

$$H_m^q f = \langle z^m \rangle \Omega[zX]\, \Omega[z^{-1}(q-1)X]^{\perp} f$$
$$= \langle z^m \rangle \Omega[zX]\, f[X + z^{-1}(q-1)]$$
$$H_m^q f\Big|_{X=1} = \langle z^m \rangle \frac{1}{1-z} f[1 + z^{-1}(q-1)]$$
$$= f[1+q-1] = f[q]$$
$$= q^{\deg f} f[1]$$

$\Omega[z] = \frac{1}{1-z}$

$\langle z^m \rangle \frac{1}{1-z} g(z^{-1}) = g(1)$ for any poly. g, any $m \geq 0$

$$H_\mu[1;q] = \left(H_{\mu_1}^q \cdots H_{\mu_\ell}^q \cdot 1 \right)_{X=1}$$
$$= q^{\mu_2 + \cdots + \mu_\ell}\, q^{n(\mu_2,\ldots,\mu_\ell)} = q^{n(\mu)}$$

$\tilde{H}_\mu(x;q)$ are characterized by:

any two of
- (o) $\tilde{H}_\mu[1;q] = 1$
- (i) $\tilde{H}_\mu(x;q) = \langle s_\lambda \mid \lambda \geq \mu \rangle$
- (ii) $\tilde{H}_\mu[X(1-q);q] \in \langle s_\lambda \mid \lambda \geq \mu^* \rangle$
- (iii) $\langle \tilde{H}_\mu, \tilde{H}_\lambda \rangle_{q,0} = 0$ for $\lambda \neq \mu$, where $\langle f,g \rangle_{q,0} = \langle f, g[X(q-1)] \rangle$

$\langle f, g[X(1-q)] \rangle_q$ ($1-q^{-1}$ ↓, $q-1$)

$H_\mu[X(1-q);q] \in \langle s_\lambda \mid \lambda \leq \mu \rangle$
$H_\mu[X(1-q^{-1});q^{-1}] \in$ ↗
$\quad \|$
$\tilde{H}_\mu[X(q-1);q]$ $\qquad \lambda^* \geq \mu^*$ ⇕
$\tilde{H}_\mu[X(1-q);q] \in \langle s_{\lambda^*} \mid \lambda \leq \mu \rangle$
$\qquad = \langle s_\lambda \mid \lambda \geq \mu^* \rangle$

Theorem There exist $\tilde{H}_\mu(x;q,t) \in \Lambda_{\mathbb{Z}[q,t]}$ s.t.

- (o) $\tilde{H}_\mu[1;q,t] = 1$
- (i) $\tilde{H}_\mu[X(1-t);q,t] = \langle s_\lambda \mid \lambda \geq \mu^* \rangle$
- (ii) $\tilde{H}_\mu[X(1-q);q,t] \in \langle s_\lambda \mid \lambda \geq \mu \rangle$
- (iii) $\langle \tilde{H}_\mu, \tilde{H}_\lambda \rangle_{q,t} = 0$ for $\lambda \neq \mu$, where $\langle f,g \rangle_{q,t} = \langle f, g[-X(1-q)(1-t)] \rangle$

Define $\tilde H_\mu(x;q,t) = \sum_\lambda \tilde K_{\lambda\mu}(q,t)$. $\left[\ \tilde K_{\lambda\mu}(q,t) \in \mathbb{N}[q,t],\quad \tilde K_{\lambda\mu}(1,1) = f_\lambda = |SYT(\lambda)|\ \right]$

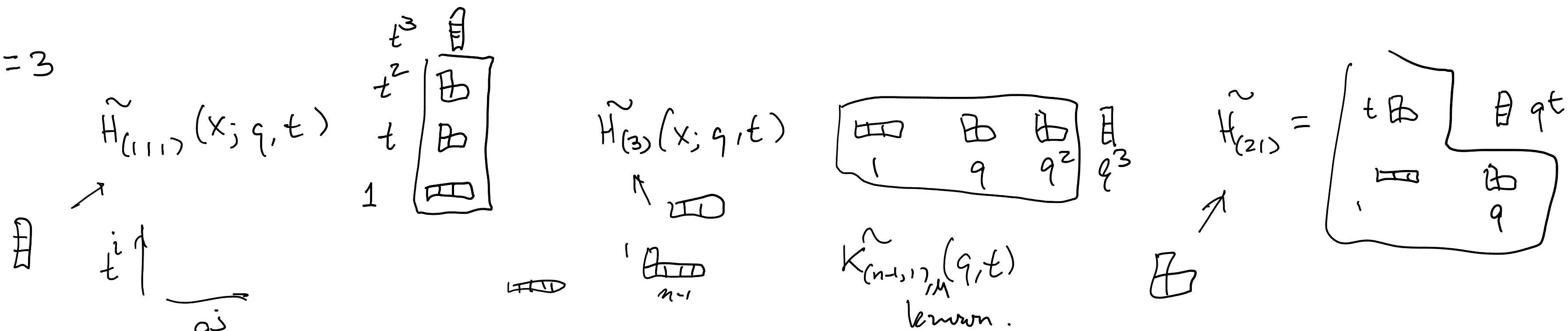
$\tilde H_\mu[X;1,1] = h_1^n$, $\quad f_\lambda = K_{\lambda,1^n}$

Ex. $\tilde H_\mu(X;0,t) = \tilde H_\mu(X;t) \leftarrow \tilde K_{\lambda\mu}(0,t) = \tilde K_{\lambda\mu}(t) = q^{n(\mu)} K_{\lambda\mu}(t^{-1})$

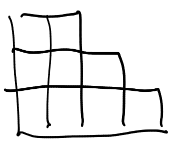
$H_{1^n} = (1-t)\cdots(1-t^n)\, h_n[X/(1-t)] = \tilde H_{1^n}(x;t)$

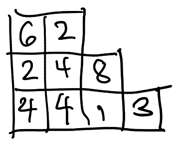
$\tilde H_{\mu^*}(X;q,t) = \tilde H_\mu(X;t,q) \qquad \tilde K_{\lambda\mu^*}(q,t) = \tilde K_{\lambda\mu}(t,q)$

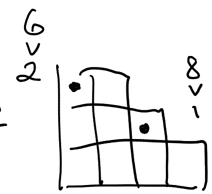
$|\mu| = 3$

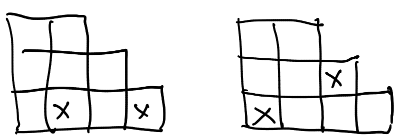


Combinatorial formula for $\tilde H_\mu(x;q,t)$

$\mu =$  $(4,3,2)$

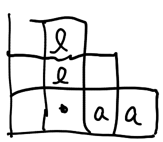
A filling $\sigma : \mu \to \mathbb{Z}_+$ 

Descent Set $D(\sigma)$ 

Attacking pairs 

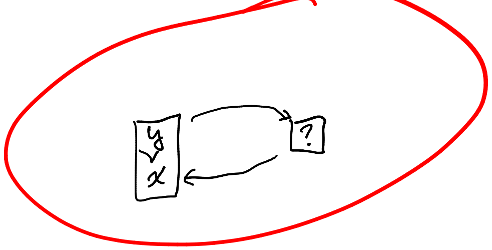
same row or consecutive rows SW–NE

Row reading word $\quad 6\ 2\ 2\ 4\ 8\ 4\ 4\ 1\ 3$ — Attacking inversions (same rows; row to row) $\quad inv(\sigma) = 7$

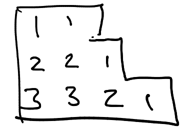
Arms + legs  arm = leg = 2

$inv'(\sigma) \overset{def}{=} inv(\sigma) - \sum_{u \in D(\sigma)} a(u) \geq 0$

$= 7 - 1 = 6$



$$\mathrm{maj}(\sigma) \underset{\text{def}}{=} \sum_{u \in D(\sigma)} (\ell+1) = 2$$

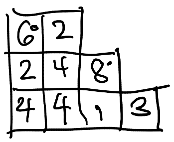


Theorem $\tilde{H}_\mu(X;q,t) = \sum_\sigma q^{\mathrm{inv}'(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$ Satisfies (i)-(iii)

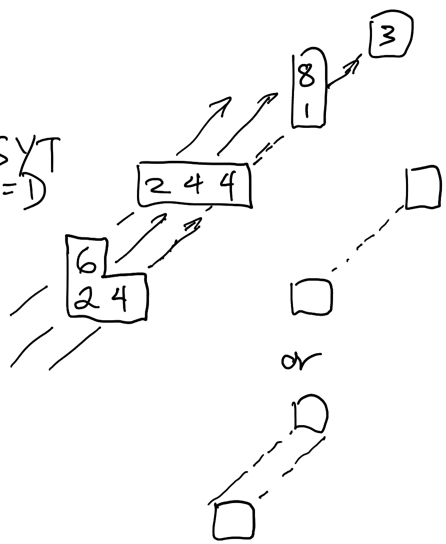
Ex. $\tilde{H}_\mu(X;1,1) = h_1^n \iff \tilde{K}_{\lambda\mu}(1,1) = K_{\lambda,1^n} = |SYT(\lambda)|$

Why is H a symmetric function?

$$F_{\mu,D} = \sum_{D(\sigma)=D} q^{\mathrm{inv}'(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma = q^{-a(D)} t^{\mathrm{maj}(D)} \sum_{D(\sigma)=D} q^{\mathrm{inv}(\sigma)} x^\sigma = q^{-a(D)} t^{\mathrm{maj}(D)} \sum_{T \in SSYT(\nu(\mu,D))} q^{\mathrm{inv}(T)} x^T$$

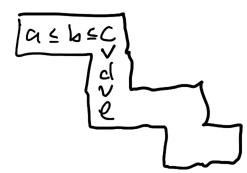
$\sigma =$  $\rightarrow$

Ribbon SSYT for $D(\sigma)=D$



attacking

or



ribbon

Given skew shapes $\underline{\nu} = (\nu^{(1)}, \dots, \nu^{(k)})$ with diagonals specified,

$$G_{\underline{\nu}}(x;q) = \sum_{T \in SSYT(\underline{\nu})} q^{\mathrm{inv}(T)} x^T$$

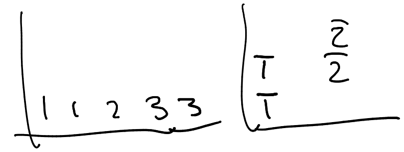
is symmetric, called an LLT polynomial, $G_{\underline{\nu}}(x;1) = \prod s_{\nu^{(i)}}(x)$, $\langle s_\lambda, G_{\underline{\nu}}(x;q)\rangle = \mathbb{N}[q]$ ← Uses KL theory

↑ not hard

$$G_{\underline{\nu}}[X - Y; q] = \sum_{T \in SSYT_{\pm}(\underline{\nu})} (-1)^{m(T)} q^{\underline{inv}(T)} x^{T_+} y^{T_-}$$

"positive" and "negative" letters

(equal − letters count as inversions)



$4_1 < 4_2 \qquad \bar{4}_2 > \bar{4}_1 \qquad 4_1 4_2 \; 4_3 \; 4_4 4_5 \qquad \bar{4}_4 \; \bar{4}_3 \; \bar{4}_2 \; \bar{4}_1$

WTS $\tilde{H}_\mu(x; q,t)$ as defined above satisfies (0), (i), (ii)

(0): Look at filling σ with all 1's $\quad q^0 t^0 = 1$

(i): is equivalent to $\tilde{H}_\mu[X(t-1); q,t] \in \langle s_\lambda \mid \lambda \le \mu \rangle$

$\in \langle m_\lambda \mid \lambda \le \mu \rangle$

$s_{\lambda^*} \mid \lambda \ge \mu^*$

$s_\lambda \mid \lambda^* \ge \mu^*$

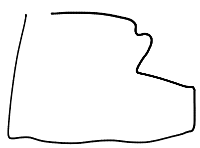
$s_\lambda \mid \lambda \le \mu$

$$\tilde{H}_\mu[\underset{\text{"}X\text{"}}{tX} - \underset{Y}{X}; q,t] = \sum_\sigma (-1)^{m(\sigma)} t^{p(\sigma)} q^{inv'(\sigma)} t^{maj(\sigma)} x^{|\sigma|}$$

want $\in \langle m_\lambda \mid \lambda \le \mu \rangle$

i.e. $\langle x^\lambda \rangle (\;) = 0$ if $\lambda \le \mu$.

Want to cancel terms for all σ with weight $\lambda \not\le \mu$.



$1 < 2 < 3 < \cdots < \bar{3} < \bar{2} < \bar{1}$

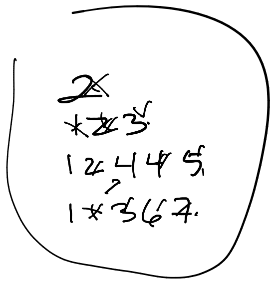
look for $\sigma(u) = x \qquad \boxed{x}_u \leftarrow i^{\text{th}}$ row $\quad |x| < i$

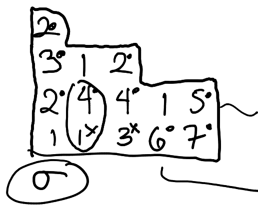
if none, leave alone. If we find one, pick specific instance, $x \longleftrightarrow \bar{x}$.

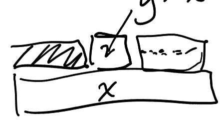
Uncancelled terms have $|\sigma(u)| \ge$ row index of u. $\quad \Rightarrow \lambda \le \mu$

(ii) .. something similar

$\tilde{H}_\mu(x;t) = \tilde{H}_\mu(x;0,t) \leftarrow q^0$ term: $\mathrm{inv}'(\sigma) = 0$







$y > x$ if possible, smallest.

otherwise, put the smallest y of all.

$y \le x$

$cw(\sigma) = 1\ 1_0\ 2\ 3_0\ 2_0\ 3_0\ 4_0\ 1_0\ 3_0\ 2\ 1\ 2_0\ 2_0\ 1_0\ 1_0$

$\ 1\ 1\ 1\ 1\ 2\ 2\ 2\ 3\ 3\ 4\ 4\ 5\ 6\ 7$

$cc(cw(\sigma)) = maj(\sigma)$

$t^{maj(\sigma)} \quad x^{\sigma}$

$\uparrow \qquad \uparrow$

$\tilde{K}_{\lambda\mu}(0,t) \quad \left(\sum_{T \in SSYT(\lambda,\mu)} t^{cc(T)}\right)\left(s_\lambda \right)$

$\Rightarrow \quad \tilde{K}_{\lambda\mu}(t) = \sum_{T \in SSYT(\lambda,\mu)} t^{cc(T)}.$